MATH2040A/B Homework 6 Solution

1 Compulsory Part

Sec. 5.1

(Sec 5.1 Q02(e)) Q: $V = P_3(\mathbb{R})$, $T(a + bx + cx^2 + dx^3) = -d + (-c + d)x + (a + b - 2c)x^2 + (-b + c - 2d)x^3$ and $\beta = \{1 - x + x^3, 1 + x^2, 1, x + x^2\}$

Ans : When written in the standard basis, we have

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -d \\ -c + d \\ a + b - 2c \\ -b + c - 2d \end{pmatrix}$$

Hence we see that

$$T\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix},\ T\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix},\ T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ T\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Hence

$$[T]_\beta = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

β is not a basis with eigenvectors.

(Sec 5.1 Q02(f)) Q: $V = M_{2\times 2}(\mathbb{R})$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{pmatrix}$
$\beta = \{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\}$

Ans : We have

$$T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = -3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, T\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, T\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, T\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

Hence all vectors in β are eigenvectors and

$$[T]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Sec 5.1 Q08) (a) Prove that a linear operator T on a finite dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

Ans :

(a) T is invertible if and only if $\det(T) \neq 0$ if and only if $\det(T - 0I) \neq 0$ if and only if 0 is not an eigenvalue of T.

(b) From (a), eigenvalues are not zero. Suffices to show one way since T, T^{-1} are inverse to each other.

$$\begin{aligned} Tv &= \lambda v \\ \frac{1}{\lambda} v &= T^{-1} v \end{aligned}$$

so λ^{-1} is an eigenvalue of T^{-1}.

(Sec 5.1 Q10) For a finite-dimensional vector space V, any scalar λ and any ordered basis β,

(a) Prove that $[\lambda \mathsf{I}_V]_\beta = \lambda I$.

(b) Compute the characteristic polynomial of $\lambda \mathsf{I}_V$.

(c) Show that $\lambda \mathsf{I}$ is diagonalizable and has only one eigenvalue.

Ans :

(a) For any $\beta_i \in \beta$, $\lambda \mathsf{I}_V \beta_i = \lambda \beta_i$, then we conclude $[\lambda \mathsf{I}_V]_\beta = \lambda I$.

(b) By definition of characteristic polynomial, $f(t) = \det(\lambda I - tI) = (\lambda - t)^n$, where n is the dimension of V.

(c) By (a) and Theorem 5.1, $\lambda \mathsf{I}$ is diagonalizable. Let $\det(\lambda I - tI) = 0$, we see the only eigenvalue is λ.

(Sec 5.1 Q20) Ans: $\det(A - tI) = f(t)$, hence $a_0 = f(0) = \det(A)$. Hence $a_0 \neq 0$ if and only if $\det(A) \neq 0$ if and only if A invertible.

Sec. 5.2

Q3(a). Let γ be the standard ordered basis of V. Then

$$[T]_\gamma = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is upper triangular and hence 0 is the only eigenvalue of T, which has multiplicity 4. However, $4 - \mathrm{rank}(T) = 1 < 4$. Therefore it is not diagonalizable.

Q3(e). Let $\beta = (\beta_1, \beta_2, \beta_3, \beta_4) = ((1,0), (0,1), (i,0), (0,i))$. Then

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

$\det(\lambda I - [T]_\beta) = (\lambda - 1)^4 + (\lambda - 1)^2 + (\lambda - 1)^2 + 1 > 0$, then it has no real eigenvalue. It is not diagonalizable.

Remark: It is also correct to consider it as a vector space over $\mathbb{C}$. Then it is diagonalizable with eigenvalues $1 + i$ and $1 - i$.

Q3(f). Let $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be an ordered basis of V. Then

$$[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of T is given by

$$\det([T]_\gamma - xI_4) = (x^2-1)(x-1)^2 = (x-1)^3(x+1).$$

It splits over $\mathbb{R}$ and the eigenvalues of T are $1, -1$, with multiplicity $3, 1$ respectively. We check that

$$[T]_\gamma - I_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence $4 - \operatorname{rank}(T - I_V) = 4 - 1 = 3$ which is the multiplicity of 1. It is clear that $\dim(E_{-1}) = 1$. Therefore T is diagonalizable.

By computation, the null space of $[T]_\gamma - I_4$ is span by the linearly independent set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Therefore

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for the eigenspace E_1.

We check that

$$[T]_\gamma + I_4 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and the null space of $[T]_\gamma + I_4$ is span by the linearly independent set $\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Therefore $\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ is a basis for the eigenspace E_{-1}.

Combining the bases, we have

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

being an ordered basis for V consisting of eigenvectors of T. Hence $[T]_\beta$ is a diagonal matrix.

Q8. The multiplicity of λ_1 is not less than $\dim(\mathsf{E}_{\lambda_1}) = n - 1$. Also, the multiplicity of λ_2 must not less than 1. Since the sum of multiplicity of λ_1 and λ_2 cannot exceed the degree of the characteristic polynomial of A, which is n, this forces the multiplicity of λ_1 being $n-1$ and that of λ_2 being 1. Hence the characteristic polynomial of A splits and multiplicity of λ_i equals to $\dim(\mathsf{E}_{\lambda_i})$ for $i = 1, 2$. So A is diagonalizable.

Q12. (a) Let $\mathsf{E}_\lambda(T) = \{v \in V : T(v) = \lambda v\}$ be the eigenspace of T associated to λ and define $\mathsf{E}_{\lambda^{-1}}(T^{-1})$ similarly.

If $v \in \mathsf{E}_\lambda(T)$, then $T(v) = \lambda v$. Apply $\lambda^{-1}T^{-1}$ on both sides, we have $\lambda^{-1}v = \lambda^{-1}T^{-1}(T(v)) = \lambda^{-1}T^{-1}(\lambda v) = T^{-1}(v)$. Therefore $v \in \mathsf{E}_{\lambda^{-1}}(T^{-1})$ and $\mathsf{E}_\lambda(T) \subset \mathsf{E}_{\lambda^{-1}}(T^{-1})$. By repeating the argument with λ^{-1} and T^{-1}, we have $\mathsf{E}_{\lambda^{-1}}(T^{-1}) \subset \mathsf{E}_{(\lambda^{-1})^{-1}}((T^{-1})^{-1}) = \mathsf{E}_\lambda(T)$. Hence we get the desired equality.

(b) If T is diagonalizable, then there exists an ordered basis β for V consisting of eigenvectors of T. By part (a), any eigenvector of T is also an eigenvector of T^{-1}. Therefore β is also consisting of eigenvectors of T^{-1} and T^{-1} is diagonalizable.

2 Optional Part

Sec. 5.1

(Sec 5.1 Q01) Ans:

(a) F, simply consider $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) T. A nonzero scalar times an eigenvector gives a new eigenvector.

(c) T, consider real matrix, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(d) F.

(e) F, see (b).

(f) F, otherwise we have infinitely many eigenvalues.

(g) F.

(h) T.

(i) T.

(j) F.

(k) F.

(Sec 5.1 Q03(c)) Ans : $\det(A - \lambda I) = -(i-\lambda)(i+\lambda) - 2$, solving gives $\lambda = 1, -1$. For $\lambda = 1$, $A \sim \begin{pmatrix} 1 & -0.5 - 0.5i \\ 0 & 0 \end{pmatrix}$ Hence an eigenvector of A is $(0.5 + 0.5i, 1)^T$.

For $\lambda = -1$, $A \sim \begin{pmatrix} 1 & 0.5 - 0.5i \\ 0 & 0 \end{pmatrix}$ hence an eigenvector of A is $(-0.5 + 0.5i, 1)^T$. Together, letting $Q := \begin{pmatrix} 0.5 + 0.5i & -0.5 + 0.5i \\ 1 & 1 \end{pmatrix}$, we have $Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(Sec 5.1 Q03(d)) Ans : $\det(A - \lambda I) = -\lambda(1-\lambda)^2$. We have eigenvalues $\lambda = 0, \lambda = 1$. For $\lambda = 0$, $A \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ hence $(1/2, 2, 1)$ is an eigenvector.

When $\lambda = 1$, $A \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, hence $(0,1,0), (1,0,1)$ are two eigenvectors.

Setting $Q = \begin{pmatrix} 1/2 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, we have $Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(Sec 5.1 Q17) Ans :

(a) Let λ an eigenvalue of T, A a corresponding eigenvector, then $A = T^2(A) = T(\lambda A) = \lambda^2 A$, $(I - \lambda^2 I)A = 0$, since $I - \lambda^2 I$ is an elemantory matrix, $0 = (I - \lambda^2 I)A = \begin{pmatrix} (1-\lambda^2)\vec{A}_1 \\ \cdots \\ (1-\lambda^2)\vec{A}_n \end{pmatrix}$, where $\vec{A}_i$ is the i-th row of A. If $\vec{A}_i$ not all zero, $(1 - \lambda^2)$ must be equal to 0, i.e., $\lambda = 1, -1$.

(b) For $\lambda = 1$, it corresponds to those matrix such that $A_{ij} = A_{ji}$, i.e., symmetric matrices. For $\lambda = -1$, it corresponds to those matrix such that $A_{ij} = -A_{ji}$, i.e., skew symmetric matrices.

(c) Take $\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$

(d) Let E_{ij} be the $n \times n$ matrix with all entries are zero except ij entry being 1, then a basis is $\{E_{ij} + E_{ji} | 1 \le i \le j \le n\} \cup \{E_{ij} - E_{ji} | 1 \le i < j \le n\}$.

(Sec 5.1 Q21) Ans:

(a) We prove by induction, for $n=2$, $\det(\begin{pmatrix} a-t & b \\ c & d-t \end{pmatrix}) = (a-t)(d-t)-bc$, so $n=2$ is true. Suppose $n-1$ is true, for the case of n, by denoting $\overline{B}_{ij}$ the cofactor matrix of B deleting i-th row j-th column, and we expand along the first row of A, we have

$$\begin{aligned}\det(A-tI) &= (A_1 1 - t)\det(\overline{A-tI}_{11}) + \sum_{i=2}^{n} A_{1i}\det(\overline{A-tI}_{1i}) \\ &= (A_{11}-t)(A_{22}-t)...(A_{nn}-t) + q'(t) + \sum_{i=2}^{n} A_{1i}\det(\overline{A-tI}_{1i})\end{aligned}$$

where in the second equality we have used the induction hypothesis, $q'(t)$ a polynomial of degree at most $n-3$, and $\sum_{i=2}^{n} A_1 i \det(\overline{A-tI}_{1i})$ is a polynomial of degree at most $n-2$, together $q(t) := q'(t) + \sum_{i=2}^{n} A_1 i \det(\overline{A-tI}_{1i})$ is a polynomial of degree at most $n-2$. Hence for the case of n it is true.

(b) $f(t) = (A_{11}-t)(A_{22}-t)...(A_{nn}-t) + q(t)$ by part (a), we have also $(A_{11}-t)(A_{22}-t)...(A_{nn}-t)$ is a polynomial of degree n, since $q(t)$ is a polynomial of degree at most $n-2$, the coefficient of t^n, t^{n-1} is the same as that of $(A_{11}-t)(A_{22}-t)...(A_{nn}-t)$, where the coefficient of t^{n-1} of $(A_{11}-t)(A_{22}-t)...(A_{nn}-t)$ is $(-1)^{n-1}tr(A)$, so $(-1)^{n-1}tr(A) = a_{n-1}$ which gives $tr(A) = (-1)^{n-1}a_{n-1}$.

Sec. 5.2

Q1. (a) False. Consider $V = \mathbb{R}^3$ and I_V.

(b) False. Consider $V = \mathbb{R}^3$, $T = I_V$ and eigenvectors $(1,0,0)$, $(2,0,0)$.

(c) False. Consider $0 \in E_\lambda$.

(d) True.

(e) True.

(f) False. Consider $V = \mathbb{R}^2$ and L_A where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(g) True.

(h) True.

(i) False. Consider $V = \mathbb{R}^2$, $W_i = \text{span}\{(1,i)\}$ for $i = 1,2,3$.

Q10. Suppose $d_i = ([T]_\beta)_{ii}$ for $i = 1,\dots,n := \dim V$. Then the characteristic polynomial of T is given by

$$\det([T]_\beta - xI_n) = \prod_{i=1}^{n}(d_i - x) \tag{1}$$

since determinants of upper triangular matrices are just product of all diagonal entries. Therefore the characteristic polynomial of T splits and each d_i is an eigenvalue. Moreover, the number of times that λ_j occurs as a diagonal entry is exactly number of times $(\lambda_j - x)$ occurs in the product equation (1), which is exactly the multiplicity m_j of λ_j for $j = 1,\dots,k$.

Q11. **Claim 1** *If C, D are $n\times n$ matrices similar to each other, then* $\text{tr}(C) = \text{tr}(D)$ *and* $\det(C) = \det(D)$.

Proof Suppose Q is an invertible matrix such that $QCQ^{-1} = D$. By Sec 2.3 Q13.,

$$\text{tr}(D) = \text{tr}(QCQ^{-1}) = \text{tr}(CQ^{-1}Q) = \text{tr}(C).$$

The assertion $\det(C) = \det(D)$ is clear. ∎

From the claim, we may assume A is upper triangular. Apply Q10. with $T = \mathsf{L}_A$ and β the standard basis, the diagonal entries of A are $\lambda_1,\dots,\lambda_k$ and that each λ_i occurs m_i times for $i = 1,\dots,k$. Therefore

$$\text{tr}(A) = \sum_{j=1}^{n} A_{jj} = \sum_{i=1}^{k} m_i\lambda_i \text{ and } \det(A) = \prod_{j=1}^{n} A_{jj} = \prod_{j=1}^{n}(\lambda_j)^{m_j}.$$

Q17. (a) Let γ be an ordered basis for V such that both $[T]_\gamma$ and $[U]_\gamma$ are diagonal matrices. Write $Q = [I_V]_\gamma^\beta$. Then Q is an invertible $n \times n$ matrix such that

$$Q^{-1}[T]_\beta Q = [I_V]_\beta^\gamma [T]_\beta [I_V]_\gamma^\beta = [T]_\gamma$$

which is diagonal. Similarly, $Q^{-1}[U]_\beta Q = [U]_\gamma$ which is also diagonal. The result follows.

(b) Suppose A and B are simultaneously diagonalizable $n \times n$ matrices. Then there exists an invertible $n \times n$ matrix Q such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices. Let β be the standard basis of $\mathbb{R}^n$. Let u_i be the i-th column vector of Q for $i = 1, \ldots, n$. Then since Q is invertible, $\gamma = \{u_1, \ldots, u_n\}$ is an ordered basis for V and $[I_V]_\gamma^\beta = Q$. Therefore

$$[\mathsf{L}_A]_\gamma = Q^{-1}[\mathsf{L}_A]_\beta Q = Q^{-1}AQ$$

which is diagonal. Similarly $[\mathsf{L}_B]_\gamma$ is also diagonal. The result follows.

Q18. (a) Let γ be an ordered basis for V such that both $[T]_\gamma$ and $[U]_\gamma$ are diagonal matrices. Then

$$[TU]_\gamma = [T]_\gamma [U]_\gamma = [U]_\gamma [T]_\gamma = [UT]_\gamma.$$

This implies $TU = UT$.

(b) Suppose A and B are simultaneously diagonalizable $n \times n$ matrices. Then there exists an invertible $n \times n$ matrix Q such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices. Then

$$AB = Q(Q^{-1}AQ)(Q^{-1}BQ)Q^{-1} = Q(Q^{-1}BQ)(Q^{-1}AQ)Q^{-1} = BA.$$